Zeta Functions in Combinatorics and Number Theory

To Prof. Keqin Feng in celebration of his 70th birthday

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Riemann zeta function

The Riemann zeta function is

\[ \zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \text{ prime}} \left(1 + p^{-s} + p^{-2s} + \cdots\right) \]

\[ = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \text{ for } \Re s > 1. \]

- \( \zeta(s) \) has a meromorphic continuation to the whole \( s \)-plane;
- It satisfies a functional equation relating \( \zeta(s) \) to \( \zeta(1 - s) \);
- It vanishes at \( s = -2, -4, \ldots \), called trivial zeros of \( \zeta \).

Riemann Hypothesis: all nontrivial zeros of \( \zeta(s) \) lie on the line of symmetry \( \Re(s) = \frac{1}{2} \).
Zeta functions of varieties

\( V: \) smooth irred. proj. variety of dim. \( d \) defined over \( \mathbb{F}_q \)

The zeta function of \( V \) counts \( N_n = \#V(\mathbb{F}_{q^n}) \):

\[
Z(V, u) = \exp \left( \sum_{n \geq 1} \frac{N_n}{n} u^n \right) = \prod_{\text{v closed pts}} \frac{1}{(1 - u^{\text{deg } v})}.
\]

When \( d = 1 \), i.e., \( V \) is a curve of genus \( g \),

\[
Z(V, u) = \prod_{1 \leq i \leq 2g} \frac{(1 - \alpha_i u)}{(1 - u)(1 - qu)}.
\]

RH: all zeros of \( Z(V; q^{-s}) \) lie on \( \Re s = \frac{1}{2} \), i.e., \( |\alpha_i| = q^{1/2} \); proved by Hasse and Weil.
For general $d$, Grothendieck proved

$$Z(V, u) = \frac{P_1(u)P_3(u) \cdots P_{2d-1}(u)}{P_0(u)P_2(u) \cdots P_{2d}(u)},$$

where $P_i(u) = \det(I - \text{Frob } u|H^i(V))$ is a polynomial in $\mathbb{Z}[u]$. RH: the roots of $P_i(u)$ have absolute value $q^{-i/2}$. Proved by Deligne.
The Ihara zeta function of a graph

- $X$ : connected undirected finite graph
- A cycle has a starting point and an orientation.
- Interested in geodesic tailless cycles.

- Two cycles are *equivalent* if one is obtained from the other by shifting the starting point.

![Figure 1: without tail](image1)
![Figure 2: with tail](image2)
• A cycle is *primitive* if it is not obtained by repeating a cycle (of shorter length) more than once.

The Ihara zeta function of $X$ counts the number $N_n$ of geodesic tailless cycles of length $n$:

$$Z(X; u) = \exp \left( \sum_{n \geq 1} \frac{N_n u^n}{n} \right) = \prod_{[C]} \frac{1}{1 - u^{l(C)}},$$

where $[C]$ runs through all equiv. classes of primitive geodesic and tailless cycles $C$, and $l(C)$ is the length of $C$. 
Properties of zeta functions of regular graphs

Ihara (1968): Let $X$ be a finite $(q+1)$-regular graph on $n$ vertices. Then its zeta function $Z(X, u)$ is a rational function of the form

$$Z(X; u) = \frac{(1 - u^2)\chi(X)}{\det(I - Au + qu^2I)},$$

where $\chi(X) = n - n(q + 1)/2 = -n(q - 1)/2$ is the Euler characteristic of $X$ and $A$ is the adjacency matrix of $X$.

If $X$ is not regular, replace $qI$ by $Q$, the degree matrix minus the identity matrix on vertices—Bass, Stark-Terras, Hoffman.
Spectral theory of regular graphs

- The trivial eigenvalues of $X$ are $\pm(q + 1)$, of multiplicity $\leq 1$. The nontrivial eigenvalues $\lambda$ satisfy $-(q + 1) < \lambda < q + 1$.

- Let $\{X_j\}$ be a family of $(q + 1)$-regular graphs with $|X_j| \to \infty$. Alon-Boppana:
  \[
  \lim_{j \to \infty} \inf \max_{\lambda \text{ of } X_j} \lambda \geq 2\sqrt{q}.
  \]

  Li, Serre: if $X_j$ contains few short cycles of odd length,
  \[
  \lim_{j \to \infty} \sup \min_{\lambda \text{ of } X_j} \lambda \leq -2\sqrt{q}.
  \]

- $[-2\sqrt{q}, 2\sqrt{q}]$ is the spectrum of the $(q + 1)$-regular tree, the universal cover of $X$. 

Ramanujan graphs and RH

- $X$ is called a *Ramanujan graph* if all $\lambda$ satisfy the bound

\[ |\lambda| \leq 2\sqrt{q}. \]

So it is spectrally optimal.

- $X$ is Ramanujan if and only if $Z(X, u)$ satisfies RH, i.e. the nontrivial poles of $Z(X, u)$ all have absolute value $q^{-1/2}$. 

The Hashimoto edge zeta function of a graph

Endow two orientations on each edge of a finite graph $X$. The neighbors of $u \to v$ are the directed edges $v \to w$ with $w \neq u$. Associate the edge adjacency matrix $A_e$.

Hashimoto (1989): $N_n = \text{Tr} A_e^n$ so that

$$Z(X, u) = \frac{1}{\det(I - A_e u)}.$$

Combined with Ihara’s Theorem, we have

$$\frac{(1 - u^2) \chi(X)}{\det(I - Au + qu^2 I)} = \frac{1}{\det(I - A_e u)}.$$
Connections with number theory

When $q = p^r$ is a prime power, let $F$ be a local field with the ring of integers $\mathcal{O}_F$ and residue field $\mathcal{O}_F/\pi\mathcal{O}_F$ of size $q$.

Eg. $F = \mathbb{Q}_p$ and $\mathcal{O}_F = \mathbb{Z}_p$, or $F = \mathbb{F}_q((x))$ and $\mathcal{O}_F = \mathbb{F}_q[[x]]$.

\[ (q + 1) - \text{regular tree} \quad \mathcal{T} = \text{PGL}_2(F)/\text{PGL}_2(\mathcal{O}_F) \]

vertices $\leftrightarrow$ PGL$_2(\mathcal{O}_F)$-cosets

vertex adjacency operator $A$ $\leftrightarrow$ Hecke operator on

\[ \text{PGL}_2(\mathcal{O}_F) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \text{PGL}_2(\mathcal{O}_F) \]

directed edges $\leftrightarrow$ $\mathcal{I}$-cosets ($\mathcal{I} =$ Iwahori subgroup)

directadjacency operator $A_e$ $\leftrightarrow$ Iwahori-Hecke operator on

\[ \mathcal{I} \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \mathcal{I} \]
\[ X = X_\Gamma = \Gamma \backslash \text{PGL}_2(F)/\text{PGL}_2(\mathcal{O}_F) = \Gamma \backslash \mathcal{T} \] for a torsion-free discrete cocompact subgroup \( \Gamma \) of \( \text{PGL}_2(F) \).

- **Ihara (1968):** A torsion-free discrete cocompact subgroup \( \Gamma \) of \( \text{PGL}_2(F) \) is free of rank \( 1 - \chi(X_\Gamma) \).

- **Take** \( F = \mathbb{Q}_p \) so that \( q = p \), \( \mathcal{O}_F = \mathbb{Z}_p \), and let \( D_\ell = \) the definite quaternion algebra over \( \mathbb{Q} \) ramified only at \( \infty \) and prime \( \ell \neq p \).

  \( \Gamma_\ell = D_\ell^\times(\mathbb{Z}[[1/p]]) \mod \text{center} \) is a discrete subgroup of \( \text{PGL}_2(\mathbb{Q}_p) \) with compact quotient; torsion-free if \( \ell \equiv 1 \pmod{12} \).

- **\( X_{\Gamma_\ell} = \Gamma_\ell \backslash \text{PGL}_2(\mathbb{Q}_p)/\text{PGL}_2(\mathbb{Z}_p) \) is a Ramanujan graph.**

- **Eichler-Shimura:** \( \frac{\det(I-Au+pu^2I)}{(1-(p+1)u+pu^2)} \) from \( X_{\Gamma_\ell} \) is the numerator of the zeta function of the modular curve \( X_0(\ell) \) mod \( p \).
Zeta functions of higher dimensional complexes

Desired properties of a combinatorial zeta function:

• it has a closed form expression giving topological and spectral information;

• it is related to zeta functions of varieties over finite fields under certain circumstances;

• it has connections with representation theory;

• it satisfies RH if and only if the underlying complex is spectrally optimal.

Consider finite-dimensional complexes arising as finite quotients of the Bruhat-Tits buildings associated to classical groups over a $p$-adic local field $F$. 

Advantages:

- The underlying group facilitates algebraic parametrizations of geometric objects.
- A building is a simply-connected simplicial complex.
- The quotient of the building by a discrete torsion-free cocompact subgroup $\Gamma$ gives a finite complex $X_\Gamma$.
- A building is a union of apartments, a geodesic in the building is a straight line in an apartment.
- A path in $X_\Gamma$ is a geodesic if it lifts to a geodesic in the building.

Will discuss zeta functions of 2-dimensional complexes which are quotients of the buildings attached to $SL_3(F)$ and $Sp_4(F)$, resp.
Figure 3: an apartment of $B_3$
The Bruhat-Tits building $\mathcal{B}_3$ attached to $SL_3(F)$

- The vertices of the building $\mathcal{B}_3$ of $SL_3(F)$ are equivalence classes of rank 3 lattices (i.e. $\mathcal{O}_F$-modules) in $F^3$.

- The group $G = PGL_3(F)$ acts transitively on vertices. Use cosets of $G$ to parametrize vertices, edges, and chambers.

- Vertices $\leftrightarrow K$ (standard maximal compact subgroup)-cosets; three types of vertices, given by $\mathbb{Z}/3\mathbb{Z}$.

- Each edge has a direction of type 1, its opposite has type 2; type one edges $\leftrightarrow E$ (parahoric subgroup) -cosets.

- Directed chambers $\leftrightarrow B$ (Iwahoric subgroup)-cosets.
Operators on $\mathcal{B}_3$

- $A_i = \text{the adjacency matrix of type } i \text{ neighbors of vertices } (i = 1, 2)$
  $A_1$ and $A_2$ are Hecke operators on certain $K$-double cosets.

- The neighbors of the type one edge $u \rightarrow v$ are the type one edges $v \rightarrow w$ such that $u, v, w$ do not form a chamber.

- $L_E = \text{the type one edge adjacency matrix.}$
  It is a parahoric operator on an $E$-double coset.
  Its transpose $L_E^t$ is the type two edge adjacency matrix.

- $L_B = \text{the adjacency matrix of directed chambers.}$
  It is an Iwahori-Hecke operator on a $B$-double coset.
Finite quotients of $B_3$

The finite complexes we consider are $X_\Gamma = \Gamma \backslash G/K = \Gamma \backslash B_3$, where $\Gamma$ is a discrete torsion-free cocompact subgroup of $G$ and $\text{ord}_\pi \text{det} \Gamma \equiv 0 \pmod{3}$ so that $\Gamma$ identifies vertices of the same type.

Division algebras of degree 3 yield many such $\Gamma$’s.

Joint work with Ming-Hsuan Kang
Zeta function of the complex $X_\Gamma$

The zeta function of $X_\Gamma$ counts the number $N_n$ of tailless geodesic cycles of length $n$ contained in the 1-skeleton of $X_\Gamma$, defined as

$$Z(X_\Gamma, u) = \exp\left(\sum_{n \geq 1} \frac{N_n u^n}{n}\right) = \prod_{[C]} \frac{1}{1 - u^{l_A(C)}},$$

where $[C]$ runs through the equiv. classes of primitive tailless geodesic cycles in the 1-skeleton of $X_\Gamma$, and $l_A(C)$ is the algebraic length of the cycle $C$.

We have

$$Z(X_\Gamma, u) = \frac{1}{\det(I - L_E u)} \frac{1}{\det(I - L^t_E u^2)}.$$

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Zeta identity for $X_\Gamma$ from $\mathcal{B}_3$

**Theorem** [Kang-L.]

(1) $Z(X_\Gamma, u)$ is a rational function given by

$$Z(X_\Gamma, u) = \frac{(1 - u^3)\chi(X_\Gamma)}{\det(I - A_1 u + qA_2u^2 - q^3u^3I) \det(I + L_Bu)},$$

where $\chi(X_\Gamma) = \#V - \#E + \#C$ is the Euler characteristic of $X_\Gamma$.

(2)

$$\frac{(1 - u^3)\chi(X_\Gamma)}{\det(I - A_1 u + qA_2u^2 - q^3u^3I)} = \frac{\det(I + L_Bu)}{\det(I - L_Eu) \det(I - L_E^t u^2)}.$$
Ramanujan complexes from $B_3$ and RH

A Ramanujan graph has its nontrivial eigenvalues fall in the spectrum of its universal cover.

The operators $A_1$ and $A_2$ on $B_3$ have the same spectrum $\Omega$.

$X_\Gamma$ is called a Ramanujan complex iff the nontrivial eigenvalues of $A_1$ and $A_2$ on $X_\Gamma$ fall in $\Omega$ iff the nontrivial zeros of $\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)$ have absolute value $q^{-1}$.

In this sense we have

(3) $X_\Gamma$ is a Ramanujan complex if and only if $Z(X_\Gamma, u)$ satisfies RH.

Ramanujan complexes are spectrally optimal since an analog of the Alon-Boppana type theorem holds, proved by Li in 2004.
Kang-L-Wang showed that the Ramanujan condition has two more equivalent statements:

(a) the nontrivial zeros of \( \text{det}(I - L_Eu) \) have absolute values \( q^{-1} \) and \( q^{-1/2} \);

(b) the nontrivial zeros of \( \text{det}(I - L_Bu) \) have absolute values \( 1, q^{-1/2}, \) and \( q^{-1/4} \).
General remarks

• There are suitable choices of $\Gamma$ so that the zeta functions of $X_\Gamma$ are related to the zeta functions of modular surfaces similar to what we saw for modular curves.

• When $\Gamma$ comes from a global congruence subgroup, $\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)$ is the local factor of a global automorphic $L$-function.

• An element in $\Gamma$ is called primitive if it generates its centralizer in $\Gamma$. 
• When $X_\Gamma$ is a graph with fundamental group $\Gamma$,

$$\frac{(1 - u^2)\chi(X_\Gamma)}{\det(I - Au + qu^2I)} = \prod_{[C]} \frac{1}{1 - u^{l_A(C)}},$$

where $[C]$ runs through conjugacy classes of primitive $C \in \Gamma$, and $l_A(C) = \text{ord}_\pi \alpha/\beta$ with $\alpha, \beta$ being the two eigenvalues of $C$ arranged so that $\alpha/\beta$ is integral.

• When $X_\Gamma$ is a complex from $\mathcal{B}_3$ with fundamental group $\Gamma$,

$$\frac{(1 - u^3)\chi(X_\Gamma)}{\det(I - A_1u + qA_2u^2 - q^3u^3I)} = \frac{1}{\det(I - LEu)} \prod_{[C]} \frac{1}{1 - u^{l_A(C)}},$$

where $[C]$ runs through conjugacy classes of primitive $C \in \Gamma$ such that $l_A(C) < l_A(C^{-1})$. 

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Figure 4: an apartment of $\Delta_4$
The building $\Delta_4$ associated to $Sp_4(F)$

Joint work with Yang Fang and Chian-Jen Wang

- The vertices of the building $\Delta_4$ of $Sp_4(F)$ are equiv. classes of certain rank 4 lattices, they fall into three kinds: primitive special, non-primitive special, and non-special. They are of type 0, 2, 3, resp.

- Up to conjugacy, $G = PGSp_4(F)$ has two maximal compact subgroups: the standard max’l compact $PGSp_4(O_F)$ and the paramodular subgroup $P_{02}$.

- $G$ acts transitively on special vertices and the special vertices $\leftrightarrow PGSp_4(O_F)$-cosets.

- There is an involution $\tau \in G$ interchanging primitive special with non-primitive special vertices.
• \( P'_{02} = \langle P_{02}, \tau \rangle \). The non-special vertices \( \leftrightarrow P'_{02}\)-cosets.

• Two kinds of edges:
  1. type 1 edges between primitive special and non-primitive special vertices
     \( \leftrightarrow E_1 \) (Siegel congruence subgroup)-cosets
  2. type 2 edges between special and non-special vertices
     \( \leftrightarrow E_2 \) (Klingen congruence subgroup)-cosets

• directed chambers \( \leftrightarrow I \) (Iwahori subgroup) - cosets

• Similarly, there are vertex adjacency operators \( A_1 \) and \( A_2 \) on special vertices, edge adjacency operators \( L_{E_1} \) and \( L_{E_2} \) on type 1 and type 2 edges, and chamber adjacency operator \( L_I \) on directed chambers. They are operators on suitable double cosets.
Zeta functions of finite quotients of $\Delta_4$

Let $\Gamma$ be a discrete torsion-free co-compact subgroup of $\text{PGSp}_4(F)$ such that $\text{ord}_\pi \det(\Gamma) \equiv 0 \pmod{4}$. Then $\Gamma$ preserves the types of the vertices. Let $X_\Gamma = \Gamma \backslash \Delta_4$. Define the zeta function of $X_\Gamma$ in the same way as the $\text{PGL}_3(F)$ case.

**Theorem** [Fang-L-Wang] The zeta function $Z(X_\Gamma, u)$ is a rational function with the following two expressions:

$$Z(X_\Gamma, u) = \frac{(1 - u^2)\chi(X_\Gamma)(1 - q^2u^2)N_s - N_{ns}}{\det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)\det(I - L_Iu)} = \frac{1}{\det(I - L_{E_1}u)\det(I - L_{E_2}u^2)},$$

where $\chi(X_\Gamma)$ is the Euler char. of $X_\Gamma$, and $N_s$ (resp. $N_{ns}$) is the number of special (resp. non-special) vertices in $X_\Gamma$. 

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Ramanujan complexes from $\Delta_4$ and RH

$X_\Gamma$ is a Ramanujan complex from $\Delta_4$

iff the infinite-dimensional representations in $L^2(\Gamma\backslash PGSp_4(F))$
containing nontrivial $PGSp_4(\mathcal{O}_F)$-invariant vectors are tempered

iff all nontrivial zeros of $\det(I - A_1 u + qA_2 u^2 - q^3 A_1 u^3 + q^6 I u^4)$
have absolute value $q^{-3/2}$

iff $Z(X_\Gamma, u)$ satisfies RH.
Like $PGL_3$ case, there are equivalent statements in terms of operators on edges of type 1, edges of type 2, and directed chambers.

**Theorem** [Fang-L-Wang] The following are equivalent:

(a) $X\Gamma$ is a Ramanujan complex from $\Delta_4$;

(b) The nontrivial zeros of $\det(I - A_1 u + qA_2 u^2 - q^3 A_1 u^3 + q^6 I u^4)$ have absolute value $q^{-3/2}$;

(c) The nontrivial zeros of $\det(I - L_{E_1} u)$ have absolute values $q^{-3/2}$ and $q^{-1}$;

(d) The nontrivial zeros of $\det(I - L_{E_2} u)$ have absolute values $q^{-3/2}$, $q^{-2}$ and $q^{-1}$;

(e) The nontrivial zeros of $\det(I - L_{I} u)$ have absolute values $q^{-3/4}$, $q^{-1}$, $q^{-1/2}$ and 1.